

INVERSE SPECTRAL PROBLEMS FOR STURM-LIOUVILLE OPERATORS WITH SINGULAR POTENTIALS, IV. POTENTIALS IN THE SOBOLEV SPACE SCALE[†]

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ABSTRACT. We solve the inverse spectral problems for the class of Sturm–Liouville operators with singular real-valued potentials from the Sobolev space $W_2^{s-1}(0, 1)$, $s \in [0, 1]$. The potential is recovered from two spectra or from the spectrum and norming constants. Necessary and sufficient conditions on the spectral data to correspond to the potential in $W_2^{s-1}(0, 1)$ are established.

1. INTRODUCTION

Suppose that q is a real-valued distribution from $W_2^{-1}(0, 1)$. We denote by (λ_n^2) and (μ_n^2) , $n \in \mathbb{N}$, eigenvalues of Sturm–Liouville operators T_D and T_N generated by the differential expression $-\frac{d^2}{dx^2} + q$ subject to the Dirichlet and Neumann–Dirichlet boundary conditions respectively (see Section 2 for precise definitions). The operators T_D and T_N are bounded below [17] and thus become positive after addition of a suitable constant to the potential q . Henceforth there is no loss of generality in assuming that the operators T_D and T_N (and the numbers λ_n and μ_n) are positive. The eigenvalues λ_n^2 and μ_n^2 are simple; we arrange them in increasing order and recall the following their properties [1, 7, 18]:

- (A1) the sequences (λ_n) and (μ_n) interlace, i.e., $\mu_n < \lambda_n < \mu_{n+1}$ for all $n \in \mathbb{N}$;
- (A2) the numbers λ_n and μ_n obey the asymptotics

$$(1.1) \quad \lambda_n = \pi n + \tilde{\lambda}_n, \quad \mu_n = \pi(n - 1/2) + \tilde{\mu}_n$$

with some ℓ_2 -sequences $(\tilde{\lambda}_n)$ and $(\tilde{\mu}_n)$.

Conversely, it was shown in [7] that if two sequences (λ_n^2) and (μ_n^2) of positive numbers satisfy properties (A1) and (A2), then there exists a unique $q \in W_2^{-1}(0, 1)$ such that (λ_n^2) and (μ_n^2) are eigenvalues of the Sturm–Liouville operators T_D and T_N with potential q . In other words, the inverse spectral problem of recovering the potential by two spectra is uniquely soluble in the class of Sturm–Liouville operators with singular potentials from $W_2^{-1}(0, 1)$. The papers [6, 7] give the corresponding reconstruction algorithm and thus extend the classical inverse spectral theory for Sturm–Liouville operators developed by Gelfand, Levitan, Marchenko, and Krein in 1950-ies [2, 11, 13].

The aim of the present work is to show that the above inverse spectral problem is completely soluble in the class of Sturm–Liouville operators with potentials from $W_2^{s-1}(0, 1)$ for every $s \in [0, 1]$. More precisely, we shall formulate necessary and sufficient conditions on sequences (λ_n^2) and (μ_n^2) in order that they are Dirichlet and

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Neumann–Dirichlet eigenvalues of a Sturm–Liouville operator with singular potential q from $W_2^{s-1}(0, 1)$.

The particular case $s = 1$ corresponds to potentials in $L_2(0, 1)$; the classical theorem by Marchenko [14, Theorem 3.4.1] states that necessary and sufficient conditions on the eigenvalues (λ_n^2) and (μ_n^2) are (A1) and (A2) with the specification that $\tilde{\lambda}_n$ and $\tilde{\mu}_n$ have the form

$$\tilde{\lambda}_n = \frac{c}{n} + \frac{\hat{\lambda}_n}{n}, \quad \tilde{\mu}_n = \frac{c}{n} + \frac{\hat{\mu}_n}{n}$$

with $c \in \mathbb{R}$ and some ℓ_2 -sequences $(\hat{\lambda}_n)$ and $(\hat{\mu}_n)$.

For an arbitrary intermediate value $s \in (0, 1)$, the direct spectral problem was studied in [8, 10, 18]. For instance, it was proved in [8] that $\tilde{\lambda}_n$ and $\tilde{\mu}_n$ are respectively even and odd sine Fourier coefficients of some function from $W_2^s(0, 1)$ (cf. the above-mentioned cases $s = 0$ and $s = 1$). More exactly, the main result from [8] reads as follows.

Theorem A. *Assume that $q \in W_2^{s-1}(0, 1)$ for some $s \in [0, 1]$ and that $\tilde{\lambda}_n, \tilde{\mu}_n$ are defined through (1.1). Then the function σ^* given by*

$$(1.2) \quad \sigma^*(x) := 2 \sum_{n=1}^{\infty} \tilde{\mu}_n \sin[(2n-1)\pi x] - 2 \sum_{n=1}^{\infty} \tilde{\lambda}_n \sin(2\pi nx)$$

belongs to $W_2^s(0, 1)$. Moreover, $\sigma^ - \sigma \in W_2^{2s}(0, 1)$, where σ is any of the distributional primitives of q .*

In the present paper we show that the condition $\sigma^* \in W_2^s(0, 1)$ is a sufficient addendum to (A1) and (A2) guaranteeing that the corresponding potential belongs to $W_2^{s-1}(0, 1)$. Our main result is as follows.

Theorem 1.1. *In order that two sequences (λ_n^2) and (μ_n^2) be eigenvalues of (positive) Sturm–Liouville operators T_D and T_N with potential from $W_2^{s-1}(0, 1)$, $s \in [0, 1]$, it is necessary and sufficient that assumptions (A1), (A2) hold and that the function σ^* of (1.2) belongs to $W_2^s(0, 1)$.*

As an intermediate step we solve the inverse spectral problem of recovering the potential of a Sturm–Liouville expression by its Dirichlet spectrum (λ_n^2) and the so-called norming constants (α_n) . We recall that

$$\alpha_n := \left(2 \int_0^1 |u_n(x)|^2 dx \right)^{-1},$$

where u_n is an eigenfunction of the operator T_D that corresponds to the eigenvalue λ_n^2 and satisfies the initial condition $u_n^{[1]}(0) = \lambda_n$, with $u^{[1]}$ denoting the quasi-derivative of a function u , see Section 2. Alternatively, we can reduce the inverse spectral problem by two spectra to recovering the potential by the spectrum (μ_n^2) of the operator T_N and the norming constants (β_n) ; the latter are defined as

$$\beta_n := \left(2 \int_0^1 |v_n(x)|^2 dx \right)^{-1},$$

where v_n is an eigenfunction of the operator T_N that corresponds to the eigenvalue μ_n^2 and satisfies the initial condition $v_n(0) = 1$.

In the case $q \in W_2^{-1}(0, 1)$ (i.e., for $s = 0$) the norming constants α_n and β_n have the asymptotics $\alpha_n = 1 + \tilde{\alpha}_n$, $\beta_n = 1 + \tilde{\beta}_n$ with ℓ_2 -sequences $(\tilde{\alpha}_n)$ and $(\tilde{\beta}_n)$, see [6].

For $s \in (0, 1]$ this asymptotics refines as follows. We introduce the function γ via the formula

$$(1.3) \quad \gamma(x) := 2 \sum_{n=1}^{\infty} \tilde{\beta}_n \cos[(2n-1)\pi x] - 2 \sum_{n=1}^{\infty} \tilde{\alpha}_n \cos(2\pi n x)$$

and also put

$$\gamma^*(x) := -2x\sigma^*(1-x).$$

Theorem 1.2. *Assume that $\sigma^* \in W_2^s(0, 1)$ for some $s \in [0, 1]$. Then the function γ also belongs to $W_2^s(0, 1)$; moreover, $\gamma - \gamma^* \in W_2^{2s}(0, 1)$.*

In particular, we see that if a primitive σ of q belongs to $W_2^s(0, 1)$, then the sequences $(-\tilde{\alpha}_n)$ and $(\tilde{\beta}_n)$ are even and odd cosine Fourier coefficients respectively of the function $\gamma \in W_2^s(0, 1)$ given by (1.3). In the reverse direction the claim is that if the even and odd parts of the functions σ^* and γ belong to $W_2^s(0, 1)$, then σ is an W_2^s -function (and hence the potential q belongs to $W_2^{s-1}(0, 1)$). More exactly, the following two statements hold true.

Theorem 1.3. *Sequences (λ_n^2) and (α_n) of positive numbers are eigenvalues and norming constants for some Sturm–Liouville operator T_D with real-valued potential $q \in W_2^{s-1}(0, 1)$, $s \in [0, 1]$, if and only if the following conditions are satisfied:*

- (B1) *the numbers $\lambda_1 < \lambda_2 < \dots$ obey the asymptotics $\lambda_n = \pi n + \tilde{\lambda}_n$, where $\tilde{\lambda}_n$ are even sine Fourier coefficients of some function from $W_2^s(0, 1)$;*
- (B2) *the numbers $\tilde{\alpha}_n := \alpha_n - 1$ are even cosine Fourier coefficients of some function from $W_2^s(0, 1)$.*

Theorem 1.4. *Sequences (μ_n^2) and (β_n) of positive numbers are eigenvalues and norming constants for some Sturm–Liouville operator T_N with real-valued potential $q \in W_2^{s-1}(0, 1)$, $s \in [0, 1]$ if and only if the following conditions are satisfied:*

- (C1) *the numbers $\mu_1 < \mu_2 < \dots$ obey the asymptotics $\mu_n = \pi(n - 1/2) + \tilde{\mu}_n$, where $\tilde{\mu}_n$ are odd sine Fourier coefficients of some function from $W_2^s(0, 1)$;*
- (C2) *the numbers $\tilde{\beta}_n := \beta_n - 1$ are odd cosine Fourier coefficients of some function from $W_2^s(0, 1)$.*

The organization of the paper is as follows. In Section 2 we give the precise definitions of the operators T_D and T_N . In Section 3 asymptotics of the norming constants is established, based on which Theorem 1.2 is proved. The algorithm of solution of the inverse spectral problems under consideration and the proofs of Theorems 1.1, 1.3, and 1.4 are given in Section 4. Finally, in Appendix A some necessary facts about Sobolev spaces $W_2^s(0, 1)$ and Fourier series therein are gathered.

2. PRELIMINARIES

Suppose that $q \in W_2^{-1}(0, 1)$ is real-valued. We fix an arbitrary real-valued distributional primitive $\sigma \in L_2(0, 1)$ of q (so that $q = \sigma'$ in the sense of distributions) and consider the differential expression

$$l_\sigma(u) := -(u' - \sigma u)' - \sigma u'$$

on its “maximal” domain in $L_2(0, 1)$,

$$\mathfrak{D}(l_\sigma) = \{u \in W_1^1(0, 1) \mid u^{[1]} \in W_1^1(0, 1), l_\sigma(u) \in L_2(0, 1)\}.$$

Here and hereafter, $u^{[1]}$ stands for the *quasi-derivative* $u' - \sigma u$ of a function u . It is easily seen that $l_\sigma(u) = -u'' + qu$ in the sense of distributions, so that l_σ is a *regularization* of the differential expression $-\frac{d^2}{dx^2} + q$.

The operators T_D and T_N are defined as the restrictions of l_σ imposing the corresponding boundary conditions

$$\begin{aligned}\mathfrak{D}(T_D) &= \{u \in \mathfrak{D}(l_\sigma) \mid u(0) = u(1) = 0\}, \\ \mathfrak{D}(T_N) &= \{u \in \mathfrak{D}(l_\sigma) \mid u^{[1]}(0) = u(1) = 0\}.\end{aligned}$$

It is known [17] that the operators T_D and T_N are selfadjoint, bounded below, and have discrete spectra. In some sense they are the most natural Dirichlet and Neumann–Dirichlet Sturm–Liouville operators associated with the differential expression $-\frac{d^2}{dx^2} + q$ with singular q ; see the discussion in [18].

3. ASYMPTOTICS OF THE NORMING CONSTANTS

Suppose that $\sigma \in L_2(0, 1)$ is real-valued. We denote by $u_\pm(\cdot, \lambda)$ and $v_\pm(\cdot, \lambda)$ solutions of the equation $l_\sigma(u) = \lambda^2 u$ that satisfy the initial conditions

$$u_-(0, \lambda) = u_+(1, \lambda) = v_-^{[1]}(0, \lambda) = 0, \quad u_-^{[1]}(0, \lambda) = u_+^{[1]}(1, \lambda) = v_-(0, \lambda) = 1.$$

Observe that according to the definition of l_σ the equation $l_\sigma(u) = \lambda^2 u$ is to be regarded as the first-order system

$$\frac{d}{dx} \begin{pmatrix} u \\ u^{[1]} \end{pmatrix} = \begin{pmatrix} \sigma & 1 \\ -\sigma^2 - \lambda^2 & -\sigma \end{pmatrix} \begin{pmatrix} u \\ u^{[1]} \end{pmatrix}.$$

Since the entries of the above matrix are integrable, this system enjoys the standard existence and uniqueness properties; in particular, the solutions $u_\pm(\cdot, \lambda)$ and $v_\pm(\cdot, \lambda)$ are well defined for all $\lambda \in \mathbb{C}$.

Set $\Phi(\lambda) := \lambda u_-(1, \lambda)$ and $\Psi(\lambda) := u_+^{[1]}(0, \lambda)$; then the numbers $\pm\lambda_n$ and $\pm\mu_n$ are zeros of Φ and Ψ respectively. Φ and Ψ are respectively odd and even entire functions of order 1 and hence can be represented by their canonical Hadamard products [7], namely,

$$\Phi(\lambda) = \lambda \prod_{n=1}^{\infty} \frac{\lambda_n^2 - \lambda^2}{\pi^2 n^2}, \quad \Psi(\lambda) = \prod_{n=1}^{\infty} \frac{\mu_n^2 - \lambda^2}{\pi^2 (n - 1/2)^2}.$$

Moreover, it turns out that the norming constants $\alpha_n = (\sqrt{2}\lambda_n \|u_-(\cdot, \lambda_n)\|)^{-2}$ and $\beta_n = (\sqrt{2}\|v_-(\cdot, \mu_n)\|)^{-2}$ can be expressed via the functions Φ and Ψ only (cf. [3, 7]).

Lemma 3.1. *The norming constants α_n and β_n satisfy the following equalities:*

$$(3.1) \quad \alpha_n = \frac{\Psi(\lambda_n)}{\dot{\Phi}(\lambda_n)}, \quad \beta_n = -\frac{\Phi(\mu_n)}{\dot{\Psi}(\mu_n)}.$$

Proof. The Green function $G_D(x, y, \lambda^2)$ of the operator T_D (i.e., the kernel of the resolvent $(T_D - \lambda^2)^{-1}$) equals

$$G_D(x, y, \lambda^2) = \sum_{n=1}^{\infty} \frac{2\alpha_n \lambda_n^2 u_-(x, \lambda_n) u_-(y, \lambda_n)}{\lambda_n^2 - \lambda^2}.$$

On the other hand, we have

$$G_D(x, y, \lambda^2) = \frac{1}{W(\lambda)} \begin{cases} u_-(x, \lambda) u_+(y, \lambda), & 0 \leq x \leq y \leq 1, \\ u_-(y, \lambda) u_+(x, \lambda), & 0 \leq y \leq x \leq 1, \end{cases}$$

where $W(\lambda) := u_+(x, \lambda)u_-^{[1]}(x, \lambda) - u_-(x, \lambda)u_+^{[1]}(x, \lambda)$ is the Wronskian of u_+ and u_- . The value of $W(\lambda)$ is independent of $x \in [0, 1]$; in particular, taking $x = 1$ and $x = 0$ we find that

$$(3.2) \quad W(\lambda) \equiv -u_-(1, \lambda) \equiv u_+(0, \lambda).$$

Equating the two expressions and comparing the residues at the poles $\lambda = \lambda_n$, we find that

$$\alpha_n \lambda_n u_-(y, \lambda_n) = \frac{u_+(y, \lambda_n)}{\dot{u}_-(1, \lambda_n)}.$$

Observe that $u_-(\cdot, \lambda_n)$ and $u_+(\cdot, \lambda_n)$ are collinear and hence

$$u_+(\cdot, \lambda_n)/u_-(\cdot, \lambda_n) = u_+^{[1]}(\cdot, \lambda_n)/u_-^{[1]}(\cdot, \lambda_n) = u_+^{[1]}(0, \lambda_n) = \Psi(\lambda_n);$$

combining the above relations, we conclude that

$$\alpha_n = \frac{u_+(y, \lambda_n)}{u_-(y, \lambda_n)} \frac{1}{\lambda_n \dot{u}_-(1, \lambda_n)} = \frac{\Psi(\lambda_n)}{\dot{\Phi}(\lambda_n)}$$

as claimed.

In a similar fashion we equate two expressions for the Green's function $G_N(x, y, \lambda^2)$ of the operator T_N , namely,

$$\sum_{n=1}^{\infty} \frac{2\beta_n v_-(x, \mu_n) v_-(y, \mu_n)}{\mu_n^2 - \lambda^2} \equiv \frac{1}{W_1(\lambda)} \begin{cases} v_-(x, \lambda) u_+(y, \lambda), & 0 \leq x \leq y \leq 1, \\ v_-(y, \lambda) u_+(x, \lambda), & 0 \leq y \leq x \leq 1. \end{cases}$$

Here W_1 is the Wronskian of u_+ and v_- and it is identically equal to $-\Psi$, as follows from the equalities

$$W_1(\lambda) := u_+(x, \lambda) v_-^{[1]}(x, \lambda) - v_-(x, \lambda) u_+^{[1]}(x, \lambda) = -u_+^{[1]}(0, \lambda) = -\Psi(\lambda).$$

Therefore we find that

$$\beta_n = \frac{\mu_n u_+(y, \mu_n)}{v_-(y, \mu_n) \dot{\Psi}(\mu_n)} = \frac{\mu_n u_+(0, \mu_n)}{\dot{\Psi}(\mu_n)} = -\frac{\mu_n u_-(1, \mu_n)}{\dot{\Psi}(\mu_n)} = -\frac{\Phi(\mu_n)}{\dot{\Psi}(\mu_n)},$$

where the second equality is obtained by taking $y = 0$, while the third one follows from (3.2). The lemma is proved. \square

In what follows, we shall say that a function $f \in L_2(0, 1)$ is *odd* (respectively *even*) if $f(1-x) \equiv -f(x)$ (respectively, if $f(1-x) \equiv f(x)$). Denote by $L_{2,o}(0, 1)$ and $L_{2,e}(0, 1)$ the subspaces of $L_2(0, 1)$ consisting of odd and even functions respectively. We shall denote by f_o and f_e respectively the odd and even parts of a function f ; obviously,

$$f_o(x) = \frac{1}{2}[f(x) - f(1-x)], \quad f_e(x) = \frac{1}{2}[f(x) + f(1-x)].$$

Lemma 3.2. *The functions Φ and Ψ admit the integral representations*

$$(3.3) \quad \begin{aligned} \Phi(\lambda) &= \sin \lambda + \int_0^1 \phi(x) \sin[\lambda(1-2x)] dx, \\ \Psi(\lambda) &= \cos \lambda + \int_0^1 \psi(x) \cos[\lambda(1-2x)] dx, \end{aligned}$$

in which $\phi \in L_{2,o}(0, 1)$ and $\psi \in L_{2,e}(0, 1)$.

Proof. Using the technique of the transformation operators [9], Φ and Ψ can be shown to admit the integral representations of the form

$$\begin{aligned}\Phi(\lambda) &= \sin \lambda + \int_0^1 \tilde{\phi}(x) \sin \lambda x \, dx, \\ \Psi(\lambda) &= \cos \lambda + \int_0^1 \tilde{\psi}(x) \cos \lambda x \, dx\end{aligned}$$

with some L_2 -functions $\tilde{\phi}$ and $\tilde{\psi}$; see detailed derivation in [7]. Now we put

$$\begin{aligned}\phi(x) &:= \begin{cases} \tilde{\phi}(1-2x) & \text{if } x \in [0, 1/2] \\ -\tilde{\phi}(2x-1) & \text{if } x \in (1/2, 1] \end{cases}, \\ \psi(x) &:= \begin{cases} \tilde{\psi}(1-2x) & \text{if } x \in [0, 1/2] \\ \tilde{\psi}(2x-1) & \text{if } x \in (1/2, 1]. \end{cases}\end{aligned}$$

It is easily seen that $\phi \in L_{2,o}(0, 1)$, $\psi \in L_{2,e}(0, 1)$, and that equalities (3.3) hold. The lemma is proved. \square

The next lemma tells us that the values of Φ and Ψ at the points λ_n and μ_n are expressed through sine and cosine Fourier coefficients of some related functions. In the following $s_n(f)$ and $c_n(f)$ will stand for respectively n -th sine and n -th cosine Fourier coefficients of a function $f \in L_2(0, 1)$; see (A.1) for exact formulae. We also denote by S the operator of multiplication by $1 - 2x$, i.e., $(Sf)(x) = (1 - 2x)f(x)$.

Lemma 3.3. *For an arbitrary $f \in L_2(0, 1)$, the following equalities hold:*

- (1) $\int_0^1 f(x) \sin[\lambda_n(1 - 2x)] \, dx = (-1)^{n+1} [s_{2n}(f) - \tilde{\lambda}_n c_{2n}(Sf) + \tilde{\lambda}_n^2 s_{2n}(f_1)];$
- (2) $\int_0^1 f(x) \cos[\lambda_n(1 - 2x)] \, dx = (-1)^n [c_{2n}(f) + \tilde{\lambda}_n s_{2n}(Sf) + \tilde{\lambda}_n^2 c_{2n}(f_2)];$
- (3) $\int_0^1 f(x) \sin[\mu_n(1 - 2x)] \, dx = (-1)^{n+1} [c_{2n-1}(f) + \tilde{\mu}_n s_{2n-1}(Sf) + \tilde{\mu}_n^2 c_{2n-1}(f_3)];$
- (4) $\int_0^1 f(x) \cos[\mu_n(1 - 2x)] \, dx = (-1)^{n+1} [s_{2n-1}(f) - \tilde{\mu}_n c_{2n-1}(Sf) + \tilde{\mu}_n^2 s_{2n-1}(f_4)],$

where f_j , $j = 1, 2, 3, 4$, are some functions from $L_2(0, 1)$.

Proof. We shall prove only part (1) as the other parts are established analogously. Using the equality

$$\begin{aligned}\sin[\lambda_n(1 - 2x)] &= (-1)^{n+1} \cos[\tilde{\lambda}_n(1 - 2x)] \sin(2\pi n x) \\ &\quad + (-1)^n \sin[\tilde{\lambda}_n(1 - 2x)] \cos(2\pi n x),\end{aligned}$$

the asymptotic relations

$$\sin t = t + O(t^3), \quad \cos t = 1 - t^2/2 + O(t^4), \quad t \rightarrow 0,$$

and the fact that $(\tilde{\lambda}_n) \in \ell_2$, we find that

$$\int_0^1 f(x) \sin[\lambda_n(1 - 2x)] \, dx = (-1)^{n+1} [s_{2n}(f) - \tilde{\lambda}_n c_{2n}(Sf) + \tilde{\lambda}_n^2 a_n]$$

for some ℓ_2 -sequence (a_n) . Clearly, there exists a function $f_1 \in L_2(0, 1)$ such that $a_n = s_{2n}(f_1)$ for all $n \in \mathbb{N}$ and the proof of part (1) is complete. \square

Remark 3.4. Put

$$\begin{aligned}g_1 &:= \sigma^* \tilde{*} Sf + \sigma^* \tilde{*} (\sigma^* \hat{*} f_1), & g_2 &:= -\sigma^* \tilde{*} Sf + \sigma^* \tilde{*} (\sigma^* \hat{*} f_2), \\ g_3 &:= \sigma^* \hat{*} Sf + \sigma^* \hat{*} (\sigma^* \tilde{*} f_3), & g_4 &:= -\sigma^* \tilde{*} Sf + \sigma^* \tilde{*} (\sigma^* \hat{*} f_4),\end{aligned}$$

where the operations $\hat{*}$ and $\tilde{*}$ are introduced in Appendix A. By virtue of Lemma A.2 we can restate equalities (1)–(4) of the previous lemma as follows:

$$\begin{aligned}
(1') \quad & \int_0^1 f(x) \sin[\lambda_n(1-2x)] dx = (-1)^{n+1} s_{2n}(f + g_1); \\
(2') \quad & \int_0^1 f(x) \cos[\lambda_n(1-2x)] dx = (-1)^n c_{2n}(f + g_2); \\
(3') \quad & \int_0^1 f(x) \sin[\mu_n(1-2x)] dx = (-1)^{n+1} c_{2n-1}(f + g_3); \\
(4') \quad & \int_0^1 f(x) \cos[\mu_n(1-2x)] dx = (-1)^{n+1} s_{2n-1}(f + g_4).
\end{aligned}$$

If the functions σ^* and f belong to $W_2^s(0,1)$ for some $s \in [0,1]$, then $Sf \in W_2^s(0,1)$ by Proposition A.1, and thus Corollary A.4 implies that the above functions g_j , $j = 1, 2, 3, 4$, belong to $W_2^{2s}(0,1)$.

Using (3.1), integral representations for Φ and Ψ , and asymptotics of λ_n and μ_n , we can show that the norming constants α_n and β_n obey the asymptotics $\alpha_n = 1 + \tilde{\alpha}_n$ and $\beta_n = 1 + \tilde{\beta}_n$ with ℓ_2 -sequences $(\tilde{\alpha}_n)$ and $(\tilde{\beta}_n)$. It turns out that if the spectral data (λ_n^2) and (μ_n^2) have better asymptotics, then the functions ϕ and ψ in (3.3) become smoother, and the asymptotics of α_n and β_n refine.

Lemma 3.5. *Assume that the numbers $\tilde{\lambda}_n := \lambda_n - \pi n$ and $\tilde{\mu}_n = \mu_n - \pi(n - 1/2)$ are such that the function σ^* of (1.2) belongs to $W_2^s(0,1)$ for some $s \in [0,1]$. Then the functions ϕ and ψ in integral representation (3.3) of Φ and Ψ have the form*

$$\phi = -\sigma_o^* + \phi_1, \quad \psi = \sigma_e^* + \psi_1,$$

where ϕ_1 and ψ_1 are respectively some odd and even functions from $W_2^{2s}(0,1)$.

Proof. In virtue of Lemma 3.2 the equality $\Phi(\lambda_n) = 0$ can be recast as

$$\sin \lambda_n + \int_0^1 \phi(x) \sin[\lambda_n(1-2x)] dx = 0.$$

Observe that $\sin \lambda_n = (-1)^n \sin \tilde{\lambda}_n$ and that $\sin \tilde{\lambda}_n = \tilde{\lambda}_n + \tilde{\lambda}_n^2 b_n$ for some ℓ_2 -sequence (b_n) . Combining this observation with Lemma 3.3, we arrive at the relation

$$\tilde{\lambda}_n - s_{2n}(\phi) + \tilde{\lambda}_n c_{2n}(S\phi) + \tilde{\lambda}_n^2 s_{2n}(\hat{\phi}) = 0$$

for some odd function $\hat{\phi} \in L_{2,o}(0,1)$. Using Lemma A.2 and recalling that $\tilde{\lambda}_n = -s_{2n}(\sigma^*) = -s_{2n}(\sigma_o^*)$, we conclude that

$$\phi = -\sigma_o^* - \sigma_o^* \hat{*} [(S\phi) - \sigma_o^* \hat{*} \hat{\phi}].$$

In particular, $\phi \in W_2^s(0,1)$ by Corollary A.4, so that the function $(S\phi) - \sigma_o^* \hat{*} \hat{\phi}$ belongs to $W_2^s(0,1)$, and again by Corollary A.4 we get $\phi_1 := \phi + \sigma_o^* = \sigma_o^* \hat{*} [\sigma_o^* \hat{*} \hat{\phi} - (S\phi)] \in W_2^{2s}(0,1)$. The fact that $\phi_1 \in L_{2,o}(0,1)$ is obvious.

In a similar manner, using the relations $\Psi(\mu_n) = 0$ and $\mu_n = \pi(n - \frac{1}{2}) + \tilde{\mu}_n$ and Lemmata 3.2 and 3.3 we find that

$$\tilde{\mu}_n - s_{2n-1}(\psi) + \tilde{\mu}_n c_{2n-1}(S\psi) + \tilde{\mu}_n^2 s_{2n-1}(\hat{\psi}) = 0$$

for some function $\hat{\psi} \in L_{2,e}(0,1)$. Replicating the above reasoning, we conclude that $\psi = \sigma_e^* + \psi_1$ for some even function ψ_1 from $W_2^{2s}(0,1)$ as claimed. The proof is complete. \square

Proof of Theorem 1.2. Formula (3.1) implies that

$$\tilde{\alpha}_n = \alpha_n - 1 = \frac{\Psi(\lambda_n) - \dot{\Phi}(\lambda_n)}{\dot{\Phi}(\lambda_n)}, \quad \tilde{\beta}_n = \beta_n - 1 = \frac{-\Phi(\mu_n) - \dot{\Psi}(\mu_n)}{\dot{\Psi}(\mu_n)}$$

for all $n \in \mathbb{N}$. According to Lemma 3.2 we have

$$\begin{aligned}\Psi(\lambda_n) - \dot{\Phi}(\lambda_n) &= \int_0^1 \theta_1(x) \cos[\lambda_n(1-2x)] dx, \\ -\Phi(\mu_n) - \dot{\Psi}(\mu_n) &= \int_0^1 \theta_2(x) \sin[\mu_n(1-2x)] dx\end{aligned}$$

with $\theta_1 := \psi - S\phi$ and $\theta_2 := -\phi + S\psi$. Further, in view of Lemma 3.3 and Remark 3.4,

$$\begin{aligned}\int_0^1 \theta_1(x) \cos[\lambda_n(1-2x)] dx &= (-1)^n c_{2n}(\theta_1 + \tilde{\theta}_1), \\ \int_0^1 \theta_2(x) \sin[\mu_n(1-2x)] dx &= (-1)^{n+1} c_{2n-1}(\theta_2 + \tilde{\theta}_2)\end{aligned}$$

with some functions $\tilde{\theta}_1$ and $\tilde{\theta}_2$ from $W_2^{2s}(0, 1)$.

It follows from Lemma 3.2 that

$$\begin{aligned}\dot{\Phi}(\lambda_n) &= \cos \lambda_n + \int_0^1 (1-2x)\phi(x) \cos[\lambda_n(1-2x)] dx, \\ -\dot{\Psi}(\mu_n) &= \sin \mu_n + \int_0^1 (1-2x)\psi(x) \sin[\mu_n(1-2x)] dx.\end{aligned}$$

We have $\cos \lambda_n = (-1)^n \cos \tilde{\lambda}_n = (-1)^n(1 + \tilde{\lambda}_n d_n)$ and $\sin \mu_n = (-1)^{n+1} \cos \tilde{\mu}_n = (-1)^{n+1}(1 + \tilde{\mu}_n e_n)$ for some ℓ_2 -sequences (d_n) and (e_n) . Using Lemma 3.3 and Remark 3.4, we now conclude that

$$(-1)^n \dot{\Phi}(\lambda_n) = 1 + c_{2n}(g_1), \quad (-1)^{n+1} \dot{\Psi}(\mu_n) = 1 + c_{2n-1}(g_2)$$

for some functions g_1 and g_2 from $W_2^s(0, 1)$. Since $\dot{\Phi}(\lambda_n) \neq 0$ and $\dot{\Psi}(\mu_n) \neq 0$ for all $n \in \mathbb{N}$, Lemma A.5 implies that

$$\frac{(-1)^n}{\dot{\Phi}(\lambda_n)} = 1 + c_{2n}(h_1), \quad \frac{(-1)^{n+1}}{\dot{\Psi}(\mu_n)} = 1 + c_{2n-1}(h_2)$$

for some functions h_1 and h_2 from $W_2^s(0, 1)$.

We now combine the above relations to conclude that

$$\tilde{\alpha}_n = c_{2n}(\theta_1 + \tilde{\theta}_1)(1 + c_{2n}(h_1)), \quad \tilde{\beta}_n = c_{2n-1}(\theta_2 + \tilde{\theta}_2)(1 + c_{2n-1}(h_2)).$$

It follows that $\gamma = -\theta_1 + \theta_2 + \tilde{\theta}$ for some $\tilde{\theta} \in W_2^{2s}(0, 1)$. Since by Lemma 3.5

$$\begin{aligned}-\theta_1 + \theta_2 &= (S - I)(\phi + \psi) = (S - I)(\sigma_e^* - \sigma_o^*) + (S - I)(\phi_1 + \psi_1) \\ &= \gamma^* + (S - I)(\phi_1 + \psi_1)\end{aligned}$$

and $(S - I)(\phi_1 + \psi_1) \in W_2^{2s}(0, 1)$ by Proposition A.1, we conclude that the function $\gamma - \gamma^*$ is in $W_2^{2s}(0, 1)$ as required. The theorem is proved. \square

Observe that Theorem 1.2 gives necessary parts of Theorems 1.3 and 1.4. Sufficient parts of these theorems constitute the inverse spectral problem and are treated in the next section.

4. THE INVERSE PROBLEM

We start by recalling briefly the standard method of recovering the potential of a Sturm–Liouville operator from the spectral data—sequences of eigenvalues and the corresponding norming constants. This method was suggested by Gelfand and Levitan in [2] for the case of regular (i.e., locally integrable) potentials and was further developed in [6] to cover singular potentials from $W_2^{-1}(0, 1)$.

Consider the functions

$$\begin{aligned}\omega_1(x) &:= \sum_{n=1}^{\infty} [\alpha_n \cos \lambda_n x - \cos(\pi n x)], & x \in [0, 2]; \\ \omega_2(x) &:= \sum_{n=1}^{\infty} \left\{ \beta_n \cos \mu_n x - \cos\left[\pi\left(n - \frac{1}{2}\right)x\right] \right\}, & x \in [0, 2],\end{aligned}$$

which belong to $L_2(0, 2)$ as soon as the functions σ^* and γ are in $L_2(0, 1)$ (cf. Lemmata 4.1 and 4.3 below) and put for $x, y \in [0, 1]$

$$(4.1) \quad f_j(x, y) := \omega_j(|x - y|) + (-1)^j \omega_j(x + y), \quad j = 1, 2.$$

We introduce an integral operator F_j with kernel f_j ; namely, F_j acts in $L_2(0, 1)$ according to the formula

$$(4.2) \quad (F_j u)(x) := \int_0^1 f_j(x, y) u(y) dy.$$

Let also K_1 and K_2 be the transformation operators for T_D and T_N respectively. Recall that K_j , $j = 1, 2$, is an integral operator with lower-triangular kernel k_j , i.e., $k_j(x, y) = 0$ a.e. on the set $\{(x, y) \mid 0 \leq x < t \leq 1\}$ and thus

$$(K_j u)(x) = \int_0^x k_j(x, y) u(y) dy, \quad j = 1, 2.$$

The operator $I + K_1$ transforms solutions of the unperturbed equation $l_0(u) = \lambda^2 u$ (i.e., corresponding to $\sigma \equiv 0$) subject to the Dirichlet initial condition $u(0) = 0$ into the solutions of the equation $l_\sigma(u) = \lambda^2 u$ subject to the Dirichlet initial condition; the operator $I + K_2$ does the same for the Neumann boundary condition at $x = 0$.

Moreover, f_j and k_j are related through the so-called Gelfand–Levitan–Marchenko (GLM) equation

$$(4.3) \quad f_j(x, y) + k_j(x, y) + \int_0^x k_j(x, t) f_j(t, y) dt = 0, \quad 0 \leq y \leq x \leq 1.$$

It is known [4, 6] that this GLM equation is naturally related to the problem of factorisation of the operator $I + F_j$ in a special manner and that (4.3) is uniquely soluble for k_j as soon as the operator $I + F_j$ is (uniformly) positive. We note that under properties (B1) and (B2) with $s = 0$ the operator $I + F_1$ is uniformly positive in $L_2(0, 1)$, and the same is true of $I + F_2$ if (C1) and (C2) hold with $s = 0$ (see the details in [6]). Henceforth in both cases of interest the GLM equation possesses a unique solution. Moreover, up to an additive constant C_j , the primitive σ of the potential q equals

$$(4.4) \quad \sigma(x) = (-1)^{j-1} 2\omega_j(2x) - 2 \int_0^x k_j(x, t) f_j(t, x) dt + C_j.$$

In what follows, we shall restrict ourselves to the case of the operator T_D ; the proofs for the operator T_N require only minor modifications.

In order to prove sufficiency part of Theorem 1.3 we need to show first that the function ω_1 belongs to $W_2^s(0, 1)$, then establish some properties of the kernel k_1 , and finally use formula (4.4) to prove the inclusion $\sigma \in W_2^s(0, 1)$.

Lemma 4.1. *Assume that the numbers $\tilde{\lambda}_n$ and $\tilde{\alpha}_n > -1$ are such that there exist functions g and h in $W_2^s(0, 1)$ with the property that $\tilde{\lambda}_n = s_{2n}(g)$ and $\tilde{\alpha}_n = c_{2n}(h)$. Then the function ω_1 belongs to $W_2^s(0, 2)$.*

Proof. Observe first that, by the construction of σ^* and γ , we have $\sigma_o^* = g_o$ and $\gamma_e = h_e$, whence $\sigma_o^* \in W_2^s(0, 1)$ and $\gamma_e \in W_2^s(0, 1)$. We write

$$\begin{aligned} 2\omega_1(2x) &= 2 \sum_{n=1}^{\infty} [(1 + \tilde{\alpha}_n) \cos(2\pi nx + 2\tilde{\lambda}_n x) - \cos(2\pi nx)] \\ &= 2 \sum_{n=1}^{\infty} \tilde{\alpha}_n \cos(2\pi nx) + 2 \sum_{n=1}^{\infty} (1 + \tilde{\alpha}_n) [\cos(2\tilde{\lambda}_n x) - 1] \cos(2\pi nx) \\ &\quad - 2 \sum_{n=1}^{\infty} (1 + \tilde{\alpha}_n) \sin(2\tilde{\lambda}_n x) \sin(2\pi nx) \\ &=: -\gamma_e(x) + g_1(x) - g_2(x), \end{aligned}$$

so that it remains to prove that the functions g_1 and g_2 belong to $W_2^s(0, 1)$.

Justification of the inclusions $g_1 \in W_2^s(0, 1)$ and $g_2 \in W_2^s(0, 1)$ is similar, and we shall give it in detail only for the function g_1 . We have

$$g_1(x) = 2 \sum_{n=1}^{\infty} (1 + \tilde{\alpha}_n) \cos(2\pi nx) \sum_{k=1}^{\infty} (-1)^k \frac{(2\tilde{\lambda}_n x)^{2k}}{(2k)!}.$$

For $x \in [0, 1]$, the estimate

$$\sum_{k=1}^{\infty} \frac{(2\tilde{\lambda}_n x)^{2k}}{(2k)!} \leq \cosh(2\tilde{\lambda}_n) - 1 = O(\tilde{\lambda}_n^2)$$

and the inclusion $(\tilde{\lambda}_n) \in \ell_2$ imply that the above double series for g_1 converges uniformly and absolutely. Changing the summation order, we find that

$$(4.5) \quad g_1(x) = \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k}}{(2k)!} x^{2k} h_k(x),$$

where

$$h_k(x) := 2 \sum_{n=1}^{\infty} (1 + \tilde{\alpha}_n) \tilde{\lambda}_n^{2k} \cos(2\pi nx).$$

Put $\tau := \sigma_o^* \tilde{*} \sigma_o^*$; then by virtue of Lemma A.2 we find that

$$h_k = \underbrace{(\tau \bar{*} \tau \bar{*} \cdots \bar{*} \tau)}_{k \text{ times}} + \gamma_e \bar{*} \underbrace{(\tau \bar{*} \tau \bar{*} \cdots \bar{*} \tau)}_{k \text{ times}};$$

see the definition of $\bar{*}$ and $\tilde{*}$ in Appendix A. Corollary A.4 yields the inclusion $h_k \in W_2^s(0, 1)$; moreover, there exists a number $C_1 > 0$ such that

$$\|h_k\|_s \leq C_1 (1 + \|\gamma_e\|_0) \|\sigma_o^*\|_s^{2k}.$$

Denote by V the operator of multiplication by x ; then V^{2k} is bounded both in $L_2(0, 1)$ and in $W_2^1(0, 1)$ and there exists $C_2 \geq 1$ such that, for all $f \in W_2^1(0, 1)$,

$$\|V^{2k} f\|_0 \leq \|f\|_0, \quad \|V^{2k} f\|_1 \leq C_2 k \|f\|_1$$

Interpolating between $W_2^1(0, 1)$ and $L_2(0, 1)$, we conclude that V^{2k} is bounded in every intermediate space $W_2^s(0, 1)$ and $\|V^{2k}f\|_s \leq C_2 k^s \|f\|_s$ for all $f \in W_2^s(0, 1)$. Combining the above relations, we conclude that the series in (4.5) converges in $W_2^s(0, 1)$. Henceforth $g_1 \in W_2^s(0, 1)$, which completes the proof. \square

Denote by \mathfrak{A}_s the set of all integral operators K over $(0, 1)$, whose kernels k possess the following properties:

- (1) for every $x \in [0, 1]$ the functions $k(x, \cdot)$ and $k(\cdot, x)$ belong to $W_2^s(0, 1)$;
- (2) the mappings

$$[0, 1] \ni x \mapsto k(x, \cdot) \in W_2^s(0, 1), \quad [0, 1] \ni x \mapsto k(\cdot, x) \in W_2^s(0, 1)$$

are continuous.

The results of [15] imply the following statement.

Proposition 4.2. *Assume that F is an integral operator with kernel f such that $F \in \mathfrak{A}_s$, $s \in [0, \frac{1}{2})$, and $I + F > 0$. Let k be equal to a solution of the corresponding GLM equation (4.3) in the domain $0 \leq y \leq x \leq 1$ and be zero in the domain $0 \leq x < y \leq 1$; then an integral operator K with kernel k also belongs to \mathfrak{A}_s .*

Let the assumptions of Lemma 4.1 hold for some $s \in [0, \frac{1}{2})$, so that $\omega_1 \in W_2^s(0, 1)$. Then the operator F_1 given by (4.1)–(4.2) belongs to \mathfrak{A}_s . Indeed, properties (1) and (2) of the definition of \mathfrak{A}_s for the kernel f_1 follow from the fact that

- (a) the operator P restricting a function on \mathbb{R} onto $(0, 1)$ is a bounded mapping from $W_2^s(\mathbb{R})$ into $W_2^s(0, 1)$ [12, Theorem 1.9.1];
- (b) the translations $T_t f(\cdot) := f(\cdot + t)$, $t \in \mathbb{R}$, form a C_0 -group in $W_2^s(\mathbb{R})$ [15].

With these preliminaries in hand, we can complete the inverse spectral analysis of Theorems 1.1, 1.3, and 1.4.

Proof of Theorem 1.1. The necessity part of the theorem follows from Theorem A, hence we need to prove only the sufficiency part, i.e., that properties (A1) and (A2) and the inclusion $\sigma^* \in W_2^s(0, 1)$ imply that $\sigma \in W_2^s(0, 1)$.

Assume therefore that the sequences (λ_n^2) and (μ_n^2) satisfy properties (A1) and (A2). Applying the reconstruction procedure explained above (and developed in detail for the case of singular potentials from $W_2^{-1}(0, 1)$ in [7]), we find a unique function $\sigma \in L_2(0, 1)$ such that λ_n^2 and μ_n^2 are eigenvalues of the corresponding operators T_D and T_N . It remains to prove that the inclusion $\sigma^* \in W_2^s(0, 1)$ yields $\sigma \in W_2^s(0, 1)$. We shall consider separately the cases $s \in [0, \frac{1}{2})$, $s \in [\frac{1}{2}, 1)$, and $s = 1$.

Case 1: $s \in [0, \frac{1}{2})$. By Theorem 1.2 the function γ belongs to $W_2^s(0, 1)$; hence by Lemma 4.1 the function ω_1 is in $W_2^s(0, 1)$ and, as explained above, the integral operator F_1 falls into the set \mathfrak{A}_s . It follows from Proposition 4.2 that the solution k_1 of the GLM equation (4.3) generates an integral operator K_1 that also belongs to \mathfrak{A}_s .

In view of formula (4.4) and the inclusion $\omega_1 \in W_2^s(0, 1)$, the theorem will be proved as soon as we show that the integral $\int_0^x k_1(x, t) f_1(t, x) dt$ defines a function from $W_2^s(0, 1)$. Observe that $k_1(x, t) = 0$ a.e. for $0 \leq x < t \leq 1$, so that we can extend the range of integration to $[0, 1]$. Hence we put

$$\eta(x) := \int_0^1 k_1(x, t) f_1(t, x) dt$$

and shall prove that $\eta \in W_2^s(0, 1)$.

Recall [12, Theorem 1.10.2] that one of the equivalent norms in the space $W_2^s(0, 1)$ is given by

$$\|\eta\|_s = \left(\|\eta\|_0^2 + 2 \int_0^1 \int_x^1 \frac{|\eta(x) - \eta(y)|^2}{(x - y)^{1+2s}} dy dx \right)^{1/2}.$$

Now we use the fact that the operators K_1 and F_1 belong to \mathfrak{A}_s . In particular, there exists a constant $C_1 > 0$ such that

$$\max_{x \in [0, 1]} (\|k_1(x, \cdot)\|_s^2 + \|k_1(\cdot, x)\|_s^2 + \|f_1(x, \cdot)\|_s^2 + \|f_1(\cdot, x)\|_s^2) \leq C_1,$$

and we use this inequality to derive the estimates

$$\|\eta\|_0^2 \leq \int_0^1 \left(\int_0^1 |k_1(x, y)|^2 dy \int_0^1 |f_1(y, x)|^2 dy \right) dx \leq C_1^2$$

and

$$\begin{aligned} |\eta(x) - \eta(y)|^2 &\leq 2 \left| \int_0^1 |k_1(x, t) - k_1(y, t)| |f_1(t, x)| dt \right|^2 \\ &\quad + 2 \left| \int_0^1 |k_1(y, t)| |f_1(t, x) - f_1(t, y)| dt \right|^2 \\ &\leq 2C_1 \left(\int_0^1 |k_1(x, t) - k_1(y, t)|^2 dt + \int_0^1 |f_1(t, x) - f_1(t, y)|^2 dt \right). \end{aligned}$$

It follows that

$$\|\eta\|_s^2 \leq C_1^2 + 4C_1 \int_0^1 \left[\|k_1(\cdot, t)\|_s^2 + \|f_1(t, \cdot)\|_s^2 \right] dt \leq 5C_1^2,$$

so that $\eta \in W_2^s(0, 1)$, and the proof for the case $s \in [0, \frac{1}{2})$ is complete.

Case 2: $s \in [\frac{1}{2}, 1)$. Case 1 applied to the exponent $\frac{s}{2}$ gives $\sigma \in W_2^{s/2}(0, 1)$, so that $\sigma - \sigma^* \in W_2^s(0, 1)$ by Theorem A. Since by assumption $\sigma^* \in W_2^s(0, 1)$, we have $\sigma \in W_2^s(0, 1)$ as required.

Case 3: $s = 1$. We again use the bootstrap method: first, by Case 2, $\sigma \in W_2^t(0, 1)$ for any $t \in [0, 1)$, e.g., $t = \frac{1}{2}$; then, by Theorem A, $\sigma - \sigma^* \in W_2^1(0, 1)$. This inclusion yields $\sigma \in W_2^1(0, 1)$, and the proof is complete. \square

Proof of Theorem 1.3. If $\sigma \in W_2^s(0, 1)$, then $\sigma^* \in W_2^s(0, 1)$ by Theorem A, hence $\gamma \in W_2^s(0, 1)$ by Theorem 1.2. Properties (B1) and (B2) then obviously hold as, by the construction of σ^* and γ , $\tilde{\lambda}_n = s_{2n}(-\sigma^*)$ and $\tilde{\alpha}_n = c_{2n}(-\gamma)$.

Conversely, assume (B1) and (B2). According to the results of [6], there exists a unique Sturm–Liouville operator T_D with potential q from $W_2^{-1}(0, 1)$ such that λ_n^2 and α_n are eigenvalues and norming constants of T_D . We need to prove that validity of (B1) and (B2) implies that the recovered potential q belongs in fact to $W_2^{s-1}(0, 1)$ (i.e., that any primitive σ of q belongs to $W_2^s(0, 1)$). In fact, under (B1) and (B2) Lemma 4.1 yields $\omega_1 \in W_2^s(0, 2)$, and it remains to observe that the proof of Theorem 1.1 derives from this the inclusion $\sigma \in W_2^s(0, 1)$. The proof is complete. \square

Proof of Theorem 1.4 is completely analogous; the only essential reservation is that Lemma 4.1 should be replaced with the following its counterpart (we leave both proofs to the reader):

Lemma 4.3. *Assume that the numbers $\tilde{\mu}_n$ and $\tilde{\beta}_n > -1$ are such that there exist functions g and h in $W_2^s(0, 1)$ with the property that $\tilde{\mu}_n = s_{2n-1}(g)$ and $\tilde{\beta}_n = c_{2n-1}(h)$. Then the function ω_2 belongs to $W_2^s(0, 1)$.*

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APPENDIX A. SOBOLEV SPACES $W_2^s(0, 1)$ AND ALL THAT

We recall here some facts about the Sobolev spaces $W_2^s(0, 1)$ and Fourier coefficients of functions from these spaces. For details, we refer the reader to [12, Ch. 1].

By definition, the space $W_2^0(0, 1)$ coincides with $L_2(0, 1)$ and the norm $\|\cdot\|_0$ in $W_2^0(0, 1)$ is just the $L_2(0, 1)$ -norm. The Sobolev space $W_2^2(0, 1)$ consists of all functions f in $L_2(0, 1)$, whose distributional derivatives f' and f'' also fall into $L_2(0, 1)$. Being endowed with the norm

$$\|f\|_2 := (\|f\|_0^2 + \|f'\|_0^2 + \|f''\|_0^2)^{1/2},$$

$W_2^2(0, 1)$ becomes a Hilbert space.

Now we interpolate [12, CH. 1.2.1] between $W_2^2(0, 1)$ and $W_2^0(0, 1)$ to get the intermediate spaces $W_2^s(0, 1)$ with norms $\|\cdot\|_s$ for $s \in (0, 2)$; namely,

$$W_2^{2s}(0, 1) := [W_2^2(0, 1), W_2^0(0, 1)]_{1-s}.$$

The norms $\|\cdot\|_s$ are nondecreasing with $s \in [0, 2]$, i.e., if $s < t$ and $f \in W_2^t(0, 1)$, then $\|f\|_s \leq \|f\|_t$. Since by construction the spaces $W_2^s(0, 1)$ form an interpolation scale, the general interpolation theorem [12, Theorem 1.5.1] implies the following interpolation property for operators in these spaces.

Proposition A.1. *Assume that an operator T acts boundedly in $W_2^s(0, 1)$ and $W_2^r(0, 1)$, $s < r$. Then T is a bounded operator in $W_2^{ts+(1-t)r}(0, 1)$ for every $t \in [0, 1]$; moreover, $\|T\|_{ts+(1-t)r} \leq \|T\|_s^t \|T\|_r^{1-t}$.*

Proposition A.1 yields boundedness in every $W_2^s(0, 1)$, $s \in [0, 2]$, of the operators R and V given by $Rf(x) = f(1-x)$ and $Vf(x) = xf(x)$.

For an arbitrary $f \in L_2(0, 1)$ and an arbitrary $\lambda \in \mathbb{C}$, we put

$$(A.1) \quad s_\lambda(f) := \int_0^1 f(x) \sin(\pi \lambda x) dx, \quad c_\lambda(f) := \int_0^1 f(x) \cos(\pi \lambda x) dx.$$

As usual, $*$ denotes the convolution operation on $(0, 1)$, i.e.,

$$(f * g)(x) := \int_0^x f(x-t)g(t) dt.$$

We shall also introduce the following shorthand notations:

$$\begin{aligned} (f \overline{*} g)(x) &:= \frac{1}{2} [R(Rf * g + f * Rg) + f * g + Rf * Rg], \\ (f \widehat{*} g)(x) &:= \frac{1}{2} [R(Rf * g + f * Rg) - f * g - Rf * Rg], \\ (f \widetilde{*} g)(x) &:= \frac{1}{2} [R(Rf * g - f * Rg) + f * g - Rf * Rg] \end{aligned}$$

(where, as earlier, R stands for the reflection operator, $(Rf)(x) = f(1-x)$). The operations $\overline{*}$, $\widehat{*}$, and $\widetilde{*}$ play the same role for the sine and cosine Fourier transform on $(0, 1)$ as the usual convolution for the Fourier transform on the whole line. Namely, these operations have the following properties.

Lemma A.2. *For arbitrary $f, g \in L_2(0, 1)$ and $\lambda \in \mathbb{C}$ the following equalities hold:*

$$c_\lambda(f)c_\lambda(g) = c_\lambda(f \overline{*} g), \quad s_\lambda(f)s_\lambda(g) = c_\lambda(f \widehat{*} g), \quad s_\lambda(f)c_\lambda(g) = s_\lambda(f \widetilde{*} g).$$

Proof. We shall prove only the first equality since the other two can be treated analogously. We have

$$2c_\lambda(f)c_\lambda(g) = \int_0^1 \int_0^1 f(x)g(t) \{ \cos[\pi\lambda(x-t)] + \cos[\pi\lambda(x+t)] \} dx dt,$$

and simple calculations lead to

$$\begin{aligned} & \int_0^1 \int_0^1 f(x)g(t) \cos \pi\lambda(x-t) dx dt \\ &= \int_0^1 \left(\int_0^{1-s} f(s+t)g(t) dt + \int_0^{1-s} f(t)g(s+t) dt \right) \cos(\pi\lambda s) ds, \\ & \int_0^1 \int_0^1 f(x)g(t) \cos[\pi\lambda(x+t)] dx dt \\ &= \int_0^1 \left(\int_0^s f(s-t)g(t) dt + \int_0^s f(1-t)g(1-s+t) dt \right) \cos \pi\lambda s ds. \end{aligned}$$

Taking into account the relations

$$\int_0^{1-s} f(s+t)g(t) dt = R(Rf * g)(s), \quad \int_0^s f(1-t)g(1-s+t) dt = Rf * Rg,$$

we get $c_\lambda(f)c_\lambda(g) = c_\lambda(f \bar{*} g)$ as stated. The lemma is proved. \square

It is well known that convolution accumulates smoothness; the precise meaning of this statement is as follows.

Proposition A.3. *Assume that $s, t \in [0, 1]$ and that $f \in W_2^s(0, 1)$ and $g \in W_2^t(0, 1)$ are arbitrary. Then the function $h := f * g$ belongs to $W_2^{s+t}(0, 1)$ and, moreover, there exists $C > 0$ independent of f and g such that $\|h\|_{s+t} \leq C\|f\|_s\|g\|_t$.*

Proof of this proposition is based on interpolation between the extreme cases $s, t = 0, 1$, which are handled with directly.

Combining Proposition A.3 with the fact that the operator R is bounded in the spaces $W_2^s(0, 1)$ for all $s \in [0, 1]$, we arrive at the following conclusion.

Corollary A.4. *Assume that $s, t \in [0, 1]$ and that $f \in W_2^s(0, 1)$, $g \in W_2^t(0, 1)$. Then the functions $f \bar{*} g$, $f \hat{*} g$, and $f \tilde{*} g$ belong to $W_2^{s+t}(0, 1)$ and, moreover, there exists a number $C > 0$ independent of f and g such that*

$$\max \{ \|f \bar{*} g\|_{s+t}, \|f \hat{*} g\|_{s+t}, \|f \tilde{*} g\|_{s+t} \} \leq C\|f\|_s\|g\|_t.$$

The following lemma is an analogue of the well-known Wiener lemma.

Lemma A.5. *Assume that $f \in W_2^s(0, 1)$, where $s \in [0, 1]$. If $1 + c_n(f) \neq 0$ for all $n \in \mathbb{N}$, then there exists a function $g \in W_2^s(0, 1)$ such that*

$$(1 + c_n(f))^{-1} = 1 + c_n(g), \quad n \in \mathbb{N}.$$

Proof. We start the proof with some auxiliary constructions. Denote by C the operator that acts from $L_2(0, 1)/\mathbb{C}$ into ℓ_2 according to the formula

$$Cf := (c_n(f))_{n \in \mathbb{N}}.$$

This operator is isomorphic; we put

$$\mathcal{W}^s := \{Cf \mid f \in W_2^s(0, 1)\}$$

and endow the linear space \mathcal{W}^s with the norm

$$\|x\|_{\mathcal{W}^s} := \|C^{-1}x\|_s.$$

In view of Lemma A.2 and Corollary A.4, the elementwise multiplication $(xy)_n := x_n y_n$ is a continuous operation in \mathcal{W}^s . We adjoin to \mathcal{W}^s the unit element e (with components e_n equal to 1) and denote the resulting unital algebra by $\widehat{\mathcal{W}}^s$. By a well-known result [16, Theorem 10.2] one can introduce an equivalent norm in $\widehat{\mathcal{W}}^s$ under which $\widehat{\mathcal{W}}^s$ becomes a commutative Banach algebra.

Assume now that the assumptions of the lemma hold and denote by x an element of $\widehat{\mathcal{W}}^s$ with components $x_n := 1 + c_n(f)$. We shall prove below that x is invertible in $\widehat{\mathcal{W}}^s$; it then follows that $x^{-1} = e + y$ for some $y \in \mathcal{W}^s$ as required.

It is well known [16, Theorem 11.5] that the element x is invertible in the unital Banach algebra $\widehat{\mathcal{W}}^s$ if and only if x does not belong to any maximal ideal of $\widehat{\mathcal{W}}^s$. Assume, on the contrary, that there exists a maximal ideal \mathfrak{m} of $\widehat{\mathcal{W}}^s$ containing x . Since $\widehat{\mathcal{W}}^s$ contains all finite sequences and none of x_n vanishes, \mathfrak{m} also contains all finite sequences. Finite sequences form a dense subset of \mathcal{W}^s because the set of all trigonometric polynomials in $\cos \pi n x$ is dense in $W_2^s(0, 1)$. Recalling that maximal ideals are closed, we conclude that $\mathcal{W}^s \subset \mathfrak{m}$. Next we observe that \mathcal{W}^s is a proper subset of \mathfrak{m} (e.g., x belongs to $\mathfrak{m} \setminus \mathcal{W}^s$) and that \mathcal{W}^s has codimension 1 in $\widehat{\mathcal{W}}^s$. Henceforth $\mathfrak{m} = \widehat{\mathcal{W}}^s$, which contradicts our assumption that \mathfrak{m} is a maximal ideal of $\widehat{\mathcal{W}}^s$. As a result, x is not contained in any maximal ideal of $\widehat{\mathcal{W}}^s$ and thus is invertible in $\widehat{\mathcal{W}}^s$. The lemma is proved. \square

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